

# ON THE PONCELET TRIANGLE CONDITION OVER FINITE FIELDS

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**ABSTRACT:** Let  $\mathbf{P}^2$  denote the projective plane over a finite field  $\mathbb{F}_q$ . A pair of nonsingular conics  $(\mathcal{A}, \mathcal{B})$  is said to satisfy the Poncelet triangle condition if, considered as conics in  $\mathbf{P}^2(\overline{\mathbb{F}}_q)$ , they intersect transversally and there exists a triangle inscribed in  $\mathcal{A}$  and circumscribed around  $\mathcal{B}$ . It is shown in this article that a randomly chosen pair of conics satisfies the triangle condition with asymptotic probability  $1/q$ . We also make a conjecture based upon computer experimentation which predicts this probability for tetragons, pentagons and so on up to enneagons.

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## 1. INTRODUCTION

1.1. We begin by recalling Poncelet's closure theorem, which is one of the most appealing results in classical projective geometry.

Let  $\mathbf{P}^2$  denote the projective plane over an algebraically closed field  $\kappa$  of characteristic not 2. Consider a pair of conics  $\mathcal{A}$  and  $\mathcal{B}$  in  $\mathbf{P}^2$  intersecting transversally. Choose a point  $P_1$  on  $\mathcal{A}$ . Draw a tangent to  $\mathcal{B}$  from  $P_1$ , intersecting  $\mathcal{A}$  again at  $P_2$ . Now repeat the construction at  $P_2$  to get a point  $P_3$  on  $\mathcal{A}$ , and then once again to get  $P_4$ . In general,  $P_4$  may not coincide with  $P_1$ ; but if it does, then  $P_1P_2P_3$  is a triangle inscribed in  $\mathcal{A}$  and circumscribed around  $\mathcal{B}$ . Such a triangle<sup>1</sup> will be called an  $\mathcal{A} \circ \mathcal{B}$  triangle.

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<sup>1</sup>In general there are two tangents to  $\mathcal{B}$  from  $P_1$ , from which we can opt for either one. If the triangle closes, then the other tangent automatically gets chosen at  $P_3$ .

Now Poncelet's theorem says that if  $P_4 = P_1$  for *some* choice of  $P_1$ , then the same is true of *any* choice of  $P_1$ . In other words, given  $\mathcal{A}$  and  $\mathcal{B}$ , the problem of constructing an  $\mathcal{A} \circ \mathcal{B}$  triangle is poristic<sup>2</sup> in the sense that, it has either no solution or infinitely many solutions (see Diagrams 1 and 2). The former case is the norm and the latter the exception. There is no such triangle if the conics are *generally* situated; that is to say, they must be in geometrically special position for the problem to be solvable.

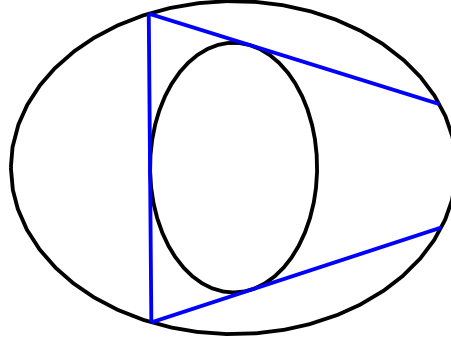


DIAGRAM 1. Conics failing the triangle condition

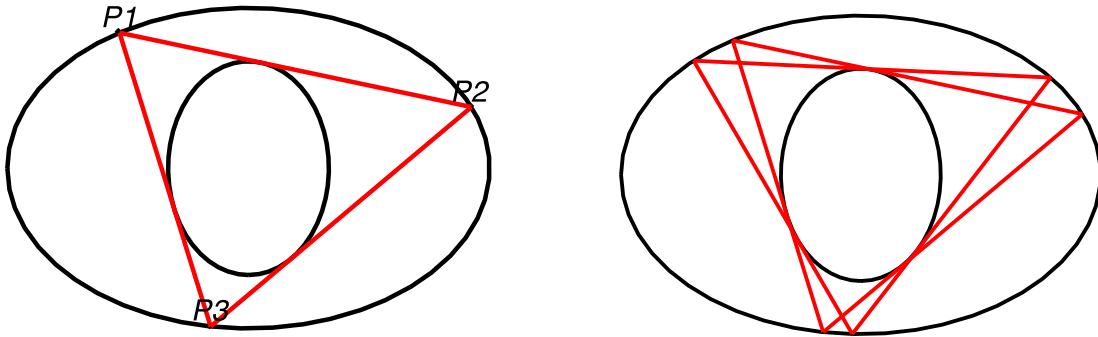


DIAGRAM 2. Conics satisfying the triangle condition

1.2. Now consider the plane  $\mathbf{P}^2(\mathbb{F}_q)$  over a finite field  $\mathbb{F}_q$ , where  $q = p^r$  and  $p \neq 2$ . A conic  $\mathcal{A} \subseteq \mathbf{P}^2(\mathbb{F}_q)$  defines a conic  $\overline{\mathcal{A}} \subseteq \mathbf{P}^2(\overline{\mathbb{F}}_q)$  given by the same equation.

**Definition 1.1.** We will say that a pair of nonsingular conics  $(\mathcal{A}, \mathcal{B})$  in  $\mathbf{P}^2(\mathbb{F}_q)$  satisfies the Poncelet triangle condition (PTC), if

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<sup>2</sup>According to John Playfair (writing in 1792): A Porism may be defined, A proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate, or capable of innumerable solutions (source: Oxford English Dictionary).

- the conics  $\overline{\mathcal{A}}$  and  $\overline{\mathcal{B}}$  intersect transversally (i.e., in four distinct points),
- There exists an  $\overline{\mathcal{A}} \circ \overline{\mathcal{B}}$  triangle.

Since (PTC) is a nontrivial geometric condition on the pair, it is natural to ask how frequently one can expect it to hold. The main result of this paper (Theorem 2.1) can be paraphrased as saying that,

The proportion of conic pairs satisfying (PTC) is asymptotically  $\frac{1}{q}$ .

In other words, the probability that a randomly chosen conic pair satisfies (PTC) is approximately  $\frac{1}{q}$ . The actual statement of the theorem gives an upper and a lower bound for this proportion.

Two clarifications are in order:

- The conics have 4 common points in  $\mathbf{P}^2(\overline{\mathbb{F}}_q)$ , and either 0, 1, 2 or all 4 of them will be in  $\mathbf{P}^2(\mathbb{F}_q)$ .
- The definition of (PTC) by itself does not require that there be an  $\mathcal{A} \circ \mathcal{B}$  triangle. However, there do exist such triangles when  $q$  is sufficiently large (see section 3.10).

1.3. Poncelet's theorem overlaps several areas of mathematics, and as such the literature associated to it is very large. The article by Bos et. al. [2] is a masterly survey of the historical development of the theorem. It contains an account of Poncelet's own proof, as well as Jacobi's proof using elliptic functions. Halbeisen and Hungerbühler [9] give another proof using Pascal's theorem. One can also find a wealth of material in the treatises by Dragović-Radnović [6] and Flatto [7]. The preprint by Hungerbühler and Kusejko [11] contains an interesting discussion of Poncelet's theorem for projective planes over prime fields. We refer the reader to Coxeter [3] and Hirschfeld [10, Ch. 7] for standard facts about conics in projective planes.

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1.4. Although a complete proof of Poncelet's theorem will not be reproduced here, we enclose a summary of the now-classic Griffiths-Harris proof [8] for the reader's interest. Assume the base field to be algebraically closed of char  $\neq 2$ , and that  $\mathcal{A}, \mathcal{B}$  intersect transversally. Let  $\mathcal{B}^* \subseteq (\mathbf{P}^2)^*$  denote the dual conic consisting of tangent lines to  $\mathcal{B}$ . Consider the subvariety  $E \subseteq \mathcal{A} \times \mathcal{B}^*$  given by

$$E = \{(P, T) : T \text{ is a tangent to } \mathcal{B} \text{ passing through } P\}.$$

The projection morphism  $E \rightarrow \mathcal{A}$  is a double cover branched over the four points in  $\mathcal{A} \cap \mathcal{B}$ . It follows by the Riemann-Hurwitz formula that  $E$  is an elliptic curve. The function  $(P_i, \overline{P_i P_{i+1}}) \rightarrow (P_{i+1}, \overline{P_{i+1} P_{i+2}})$ , from section 1.1 corresponds to a translation

$$E \rightarrow E, \quad z \rightarrow z + \tau,$$

by some constant  $\tau \in E$ . Now  $P_4 = P_1$ , iff  $\tau$  is a 3-torsion point of  $E$ . But  $\tau$  depends only on the relative positions of  $\mathcal{A}$  and  $\mathcal{B}$ , and hence  $P_4 = P_1$  is true either for no  $P_1$  or for all  $P_1$ .  $\square$

The argument remains unchanged if 3 is replaced by any  $n$ . Thus, if there exists an  $n$ -gon inscribed in  $\mathcal{A}$  and circumscribed around  $\mathcal{B}$ , then there exists one starting from any point in  $\mathcal{A}$ . Although the main result of this paper applies only to triangles, we propose a conjecture about the next few values of  $n$  (see section 4).

1.5. Even if the pair  $(\mathcal{A}, \mathcal{B})$  satisfies (PTC), it may happen over a finite field that no tangent can be drawn to  $\mathcal{B}$  from some choices of  $P_1$  in  $\mathcal{A}$ . (This will be the case if the polar line of  $P_1$  with respect to  $\mathcal{B}$  does not intersect  $\mathcal{B}$  in an  $\mathbb{F}_q$ -rational point.) However, if such a tangent does exist, then one can complete an  $\mathcal{A} \circ \mathcal{B}$  triangle. Examples of either phenomenon will be given in section 2.3.

## 2. THE MAIN THEOREM

Assume that  $\text{char}(q) \neq 2, 3$ . Let  $\Psi$  denote the set of conic pairs  $(\mathcal{A}, \mathcal{B})$  in  $\mathbf{P}^2(\mathbb{F}_q)$ , such that  $\overline{\mathcal{A}}, \overline{\mathcal{B}}$  intersect transversally. Let  $\Gamma$  denote the subset of pairs satisfying (PTC).

**Theorem 2.1.** With notation as above,

$$\frac{q-16}{q(q+1)} \leq \frac{|\Gamma|}{|\Psi|} \leq \frac{q+5}{(q-2)(q-3)}.$$

One can think of a conic pair in  $\Psi$  as being a candidate for satisfying (PTC). According to the theorem, the probability that it actually does so is  $\frac{1}{q} + O\left(\frac{1}{q^2}\right)$ .

2.1. Our main tool will be an algebraic criterion due to Cayley for (PTC) to hold (see [8]). Let  $[x, y, z]$  be homogeneous coordinates in  $\mathbf{P}^2$ . Let  $\mathcal{A}, \mathcal{B}$ , which are assumed to intersect transversally in  $\mathbf{P}^2(\overline{\mathbb{F}}_q)$ , respectively correspond to symmetric  $3 \times 3$  matrices  $A, B$ . Write  $\Delta = \det(tA + B)$ , where  $t$  is an indeterminate. Now consider a formal Maclaurin series expansion

$$\sqrt{\Delta} = H_0 + H_1 t + H_2 t^2 + \dots,$$

where the  $H_i$  are functions of entries in  $A$  and  $B$ . Then we have the following criterion:

**Proposition 2.2** (Cayley). The pair  $(\mathcal{A}, \mathcal{B})$  satisfies (PTC), if and only if  $H_2 = 0$ .

In the context of the proof-sketch above,  $E$  has an affine model given by the equation  $u^2 = \Delta$  in the variables  $t, u$ . Now the criterion is proved by an explicit calculation which detects the inflection points of  $E$  (see [loc. cit.]).

**2.2. A sample calculation.** We will begin by determining the (PTC)-pairs in a special case. Most of the ideas needed for the main theorem are already present in this calculation. For simplicity, assume that  $q$  is a prime number  $\geq 7$ . Consider the pencil of conics in  $\mathbf{P}^2(\mathbb{F}_q)$  passing through the points  $[1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1]$ . Each nonsingular conic in the pencil may be written as

$$C_\alpha : \alpha xy + (1 - \alpha)xz - yz = 0,$$

for some  $\alpha \in \mathbb{F}_q \setminus \{0, 1\}$ . It corresponds to the symmetric matrix  $\begin{bmatrix} 0 & \alpha & 1 - \alpha \\ \alpha & 0 & -1 \\ 1 - \alpha & -1 & 0 \end{bmatrix}$ .

Let  $\mathcal{A} = C_r$  and  $\mathcal{B} = C_s$  for some  $r, s \neq 0, 1$ ; the number of such pairs is  $(q - 2)(q - 3)$ . Now consider the subset

$$\mathfrak{P} = \{(r, s) : H_2(r, s) = 0, \text{ and } r, s \neq 0, 1\},$$

of conics satisfying (PTC). We will determine the size of  $\mathfrak{P}$ .

A straightforward calculation shows that  $\Delta = (rt + s)(rt + s - t - 1)(t + 1)$ , and<sup>3</sup>

$$(1) \quad H_2(r, s) = r^2 + (6s^2 - 4s^3 - 4s)r + s^4.$$

Considered as a quadratic in  $r$ , its discriminant is

$$(2) \quad \delta = (6s^2 - 4s^3 - 4s)^2 - 4s^4 = \underbrace{16s^2(s-1)^2}_{e(s)} \times \underbrace{(s^2 - s + 1)}_{f(s)}.$$

Thus, for a given value of  $s$ , the equation  $H_2(r, s) = 0$  has one root in  $r$  if  $f(s) = 0$ , two distinct roots if  $f(s)$  is a nonzero square in  $\mathbb{F}_q$ , and no roots otherwise.

**Claim-1:** The set  $\mathfrak{P}$  has cardinality  $q - 5$ .

Given claim-1, the proportion of (PTC)-pairs in the pencil is

$$\frac{q - 5}{(q - 2)(q - 3)} \simeq \frac{1}{q}.$$

To prove claim-1, consider the set  $S = \{s \in \mathbb{F}_q \setminus \{0, 1\} : f(s) \text{ is a nonzero square}\}$ . Let  $\left(\frac{m}{q}\right)$  denote the Legendre symbol.

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<sup>3</sup>The denominator of  $H_2$  is  $[s(s - 1)]^{\frac{3}{2}}$ , which can be ignored.

**Claim-2:** The size of  $S$  is given by

$$|S| = \begin{cases} \frac{q-7}{2} & \text{if } \left(\frac{-3}{q}\right) = +1, \\ \frac{q-5}{2} & \text{if } \left(\frac{-3}{q}\right) = -1. \end{cases}$$

Claim-1 follows from claim-2. Indeed, suppose that  $\left(\frac{-3}{q}\right) = +1$ . Then  $S$  contributes  $2 \times \frac{q-7}{2} = q-7$  pairs to  $\mathfrak{P}$ . Furthermore,  $f(s) = (s - 1/2)^2 + 3/4 = 0$  has two roots in  $\mathbb{F}_q$ , each of which contributes one pair, making up the total of  $q-5$ . The other case follows by a similar argument.

To prove claim-2, write  $(s - 1/2)^2 + 3/4 = y^2$  for  $s \in S$ . From

$$\underbrace{(y - s + 1/2)}_a \underbrace{(y + s - 1/2)}_{3/4a} = 3/4,$$

we get  $s = \frac{3+4a-4a^2}{8a}$  for some  $a \neq 0$ . Since  $s \neq 0, 1$ , we must have  $a \neq \pm 1/2, \pm 3/2$ . Altogether this excludes five values of  $a$ . If  $\frac{1}{2}(1 \pm \sqrt{-3}) \in \mathbb{F}_q$ , then two more values (namely  $a = \pm \frac{\sqrt{-3}}{2}$ ) are excluded, since  $f(s) = 0$  is disallowed. Since  $a$  and  $-3/4a$  lead to the same  $s$ -value, we must divide the number of possible  $a$ -values by 2. Thus we get either  $\frac{q-7}{2}$  or  $\frac{q-5}{2}$ , according to whether  $\sqrt{-3}$  does or does not belong to<sup>4</sup>  $\mathbb{F}_q$ .  $\square$

**2.3. Example.** Let  $q = 43$ ,  $\mathcal{A} = C_{11}$  and  $\mathcal{B} = C_{36}$ . It is easy to verify that  $H_2 = 0$ , so that (PTC) holds. Let  $P_1 = [1, 17, 34]$  on  $\mathcal{A}$ . The polar line to  $\mathcal{B}$  with respect to  $P_1$  is  $x + 18y + 5z = 0$ . It intersects  $\mathcal{B}$  in the two points  $R_1 = [1, 32, 5]$  and  $R'_1 = [1, 40, 2]$ . If we choose  $R_1$ , then  $P_1 R_1$  is a tangent to  $\mathcal{B}$  which intersects  $\mathcal{A}$  again at  $P_2 = [1, 36, 3]$ . Repeating the construction at  $P_2$  leads to  $P_3 = [1, 24, 28]$ , and then back to  $P_1$ . This gives an  $\mathcal{A} \circ \mathcal{B}$  triangle. By contrast, if  $P_1 = [1, 9, 12]$ , its polar line with respect to  $\mathcal{B}$  is  $x + 32y + 13z = 0$ , which does not intersect  $\mathcal{B}$ . The construction cannot proceed any further, and there is no  $\mathcal{A} \circ \mathcal{B}$  triangle with  $[1, 9, 12]$  as a vertex. (Of course, there will exist an  $\overline{\mathcal{A}} \circ \overline{\mathcal{B}}$  triangle.)

A choice of  $P_1$  on  $\mathcal{A} \cap \mathcal{B}$  will lead to a degenerate triangle, with two coincident vertices. For instance, with the same conics as above, let  $P_1 = [0, 1, 0]$ . The tangent to  $\mathcal{B}$  through  $P_1$  intersects  $\mathcal{A}$  again at  $P_2 = [1, 20, 36]$ . Now the tangent to  $\mathcal{B}$  through  $P_2$  is  $x + 14y + 34z = 0$ , which is also the tangent to  $\mathcal{A}$  at  $P_2$ . Hence  $P_3$  coincides with  $P_2$ .

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<sup>4</sup>This condition can be made explicit using the quadratic reciprocity theorem (see [12, Ch. 5]). We have  $\left(\frac{-3}{q}\right) = +1$  (resp.  $-1$ ) if  $q \equiv 1, 7$  (resp.  $5, 11$ ) mod 12.

2.4. It is easy to check directly that

$$H_2(1-r, 1-s) = H_2(r, s), \quad e(1-s) = e(s), \quad f(1-s) = f(s).$$

In geometric language, interchanging  $y$  and  $z$  defines an involution on the pencil which interchanges  $C_\alpha$  and  $C_{1-\alpha}$ . This  $\mathbf{Z}_2$ -invariance will play a small role later.

### 3. THE DICKSON CLASSIFICATION

In this section we will complete the proof of the main theorem. In outline, the strategy is to decompose  $\Psi$  into a union of pencils, and estimate the proportion of (PTC)-pairs in each pencil.

3.1. Two quadratic forms  $F(x, y, z), G(x, y, z)$  form a pencil

$$\pi = \{\eta F + G = 0 : \eta \in \mathbf{P}^1(\mathbb{F}_q)\} \subseteq \mathbf{P}^2(\mathbb{F}_q).$$

By convention,  $\eta = \infty$  corresponds to  $F = 0$ . All such pencils have been classified up to projective automorphisms by Dickson [5]; this classification is also described in a table on page 175 of Hirschfeld [10]. There are altogether 20 isomorphism classes, but 15 of them are fortunately disqualified for at least one of the following reasons:

- Every member of the pencil is singular (e.g., (1)-st entry in the table).
- The generators of the pencil do not intersect transversally in  $\mathbf{P}^2(\overline{\mathbb{F}}_q)$  (e.g., (4)-th entry).
- The pencil can only occur in characteristic 2 (e.g., (7)-th entry).

This leaves five isomorphism classes, to be called class  $(i)$  for  $i = 3, 14, 16, 18, 19$ , corresponding to their positions in the table.

For class (3), the generators are  $F = xy, G = z^2 + yz + xz$ . They intersect in four  $\mathbb{F}_q$ -rational points, namely  $[0, 1, 0], [0, 1, -1], [1, 0, 0], [1, 0, -1]$ . The pencil in section 2.2 is of this class.

For class (14), the generators are

$$F = xy, \quad G = y^2 + yz + xz + e z^2,$$

where  $e \in \mathbb{F}_q$  is any element such that the polynomial  $T^2 + T + e$  is irreducible over  $\mathbb{F}_q$ .

For class (16), the generators are  $F = xy, G = e_1 x^2 + e_2 y^2 + xz + yz + z^2$ , where  $e_1, e_2$  are such that  $T^2 + T + e_1, T^2 + T + e_2$  are irreducible.

For class (18), the generators are  $F = y^2 - xz, G = x^2 + b y^2 + c xy + yz$ , where  $b, c$  are such that

$$(3) \quad g(T) = T^3 + bT^2 + cT + 1,$$

is irreducible.

For class (19), the generators are  $F = x^2 - \nu y^2, G = z^2 - \rho y^2 + 2\sigma xy$ , where  $\nu$  and  $\rho^2 - 4\nu\sigma^2$  are non-squares.

Notice that  $F = 0$  is a singular conic for all classes except (18). Each of the other four classes has three singular members corresponding to  $\eta_1, \eta_2, \eta_3$ , where  $\eta_1, \eta_2 \in \overline{\mathbb{F}}_q$  are the roots of the quadratic equation  $\text{discrim}(\eta F + G) = 0$ , and  $\eta_3 = \infty$ . For class (18), this equation is a cubic in  $\eta$ , which is in fact identical to  $g(\eta) = 0$ . Because of this small anomaly, one will have to make a separate argument for class (18).

3.2. We will say that a pencil  $\pi$  is *eligible* if it belongs to any of these five isomorphism classes. Let  $\Psi_\pi$  denote the set of pairs of nonsingular conics in  $\pi$ , and let  $\Gamma_\pi$  denote the subset of pairs satisfying (PTC). Write

$$\mathbb{L} = \frac{q-16}{q(q+1)}, \quad \mathbb{U} = \frac{q+5}{(q-2)(q-3)}.$$

for the lower and upper bounds in the main theorem.

**Proposition 3.1.** For every eligible pencil  $\pi$ , we have

$$\mathbb{L} \leq \frac{|\Gamma_\pi|}{|\Psi_\pi|} \leq \mathbb{U}.$$

Given the proposition, the main theorem follows immediately. We have decompositions  $\Psi = \bigcup_{\pi} \Psi_\pi$  and  $\Gamma = \bigcup_{\pi} \Gamma_\pi$ , quantified over eligible pencils. Then

$$\mathbb{L} \leq \frac{|\Gamma|}{|\Psi|} = \frac{\sum_{\pi} |\Gamma_\pi|}{\sum_{\pi} |\Psi_\pi|} \leq \mathbb{U}.$$

Here we have used the elementary inequality

$$\min_i \left\{ \frac{a_i}{b_i} \right\} \leq \frac{a_1 + a_2 + \dots}{b_1 + b_2 + \dots} \leq \max_i \left\{ \frac{a_i}{b_i} \right\}.$$

□

3.3. The central idea behind the proposition is that the structure of  $H_2$  and  $\delta$  for any eligible pencil is similar to the one in the sample calculation, which allows us to make a qualitative estimate along the lines of claim-2. We will break down the argument in a couple of lemmas. Let  $\mathcal{A}, \mathcal{B}$  respectively correspond to  $A = rF + G, B = sF + G$ .

**Lemma 3.2.** Assume that  $\pi$  is of any class except (18). Then

- The polynomial  $H_{2,\pi}(r, s)$  is of degree 2 in  $r$ , and degree 3 in  $s$ .



- Its  $r$ -discriminant  $\delta_\pi$  is of the form  $e_\pi(s) \times f_\pi(s)$ , where  $e_\pi(s)$  is the square of a quadratic polynomial, and  $f_\pi(s)$  is a quadratic polynomial which is not the square of a linear polynomial.

For instance, for the class (14) pencil,

$$\delta_\pi = \underbrace{16(e s^2 - s + 1)^2}_{e_\pi(s)} \times \underbrace{(e^2 s^2 - e s - 3e + 1)}_{f_\pi(s)}.$$

If  $f_\pi(s)$  were to be the square of a linear polynomial, its  $s$ -discriminant

$$e^2 - 4e^2(1 - 3e) = 3e^2(4e - 1) = 0,$$

implying that  $e = 0$  or  $\frac{1}{4}$ . But then  $T^2 + T + e$  cannot be irreducible in either case.

PROOF. Let  $\bar{\pi}$  denote the corresponding pencil (defined by the same generators) in  $\mathbf{P}^2(\bar{\mathbb{F}}_q)$ . Its base locus consists of a quadruple of non-collinear points, and any two such quadruples can be taken to each other via an automorphism of  $\mathbf{P}^2(\bar{\mathbb{F}}_q)$ . Thus all such pencils are isomorphic over  $\bar{\mathbb{F}}_q$ , and we can obtain  $H_{2,\pi}$  and  $\delta_\pi$  by transforming the corresponding expressions (1), (2) from section 2.2. Since the singular members must correspond, the two pencils are related by the affine substitution  $\alpha = \frac{\eta - \eta_1}{\eta_2 - \eta_1}$ , so that  $\eta = \eta_1, \eta_2, \infty$  respectively map to  $\alpha = 0, 1, \infty$ . But then the same substitution on  $r, s$  transforms  $H_{2,e}(s)$  and  $f(s)$  into  $H_{2,\pi,e_\pi}$  and  $f_\pi$ . Since all degrees are preserved, and the property of being a square or a non-square is likewise preserved, we have the result.  $\square$

The coefficients of the substitution are in  $\bar{\mathbb{F}}_q$ , and not necessarily in  $\mathbb{F}_q$ . However, notice that  $1 - \alpha = \frac{\eta - \eta_2}{\eta_1 - \eta_2}$ , which is  $\alpha$  with  $\eta_1, \eta_2$  interchanged. Now the invariance in section 2.4 implies that the coefficients of  $H_{2,\pi,e_\pi}, f_\pi$  are symmetric in  $\eta_1, \eta_2$ , and hence lie in  $\mathbb{F}_q$ . Such an argument will not work on class (18), since one would need a fractional linear transformation to move  $\alpha = \infty$  to a finite point  $\eta_i$ .

3.4. We can now estimate the size of  $\Gamma_\pi$ . The idea, as before, is to consider how often  $f_\pi(s)$  is a square. Let  $\varphi(s) \in \mathbb{F}_q[s]$  be any quadratic polynomial which is not the square of a linear polynomial, and let

$$\mathcal{Z}_\varphi = \{s \in \mathbb{F}_q : \varphi(s) \text{ is a square in } \mathbb{F}_q\}.$$

**Lemma 3.3.** With notation as above,

$$\frac{q-1}{2} \leq |\mathcal{Z}_\varphi| \leq \frac{q+5}{2}.$$

PROOF. We will be brief, since the argument is similar to claim-2. (But the values  $s = 0, 1$  are no longer disallowed.) Say  $\varphi(s) = u_0 s^2 + u_1 s + u_2$ . If  $u_0$  is a square, then dividing by it leaves  $\mathcal{Z}$  unchanged, hence we may write  $f(s) = (s - b)^2 + c$  for some  $c \neq 0$ . Then, as in claim-2, each element in  $\mathcal{Z}_\varphi$  is of the form  $s = \frac{c+2ba-a^2}{2a}$  for some  $a \neq 0$ . Now  $a, -c/a$  lead to the same  $s$ -value, and hence we get  $|\mathcal{Z}_\varphi| = \frac{q+1}{2}$  or  $\frac{q-1}{2}$  depending on whether  $\sqrt{-c}$  does or does not belong to  $\mathbb{F}_q$ . In any case,

$$\frac{q-1}{2} \leq |\mathcal{Z}_\varphi| \leq \frac{q+1}{2}.$$

If  $u_0$  is not a square, then consider  $\tilde{\varphi}(s) = \varphi(s)/u_0$ . Then  $\varphi(s)$  is a square iff  $\tilde{\varphi}(s)$  is a non-square, unless they are both zero. Applying the earlier estimate to  $\tilde{\varphi}$  and taking complements, we get

$$\frac{q-1}{2} \leq |\mathcal{Z}_\varphi| \leq \frac{q+5}{2}.$$

This proves the lemma. □

3.5. Now let  $\pi$  be an eligible pencil, not of class (18), with singular members  $\eta_1, \eta_2$ . Depending on its structure, it may happen that both  $\eta_i$  belong to  $\mathbb{F}_q$  or neither of them does.

Since an element in  $\mathcal{Z}_{f_\pi}$  can contribute at most two pairs to  $\Gamma_\pi$ , we have  $|\Gamma_\pi| \leq 2|\mathcal{Z}_{f_\pi}|$ . It remains to find a lower bound. We get only one  $r$ -value if  $f_\pi(s) = 0$ . Since there are at most two roots of  $f_\pi(s)$  in  $\mathbb{F}_q$ , this means a loss of at most two pairs. Moreover, at most  $2 \times 2 = 4$  pairs may be lost because either  $r$  or  $s$  equals  $\eta_i$ . Thus  $|\Gamma_\pi| \geq 2|\mathcal{Z}_{f_\pi}| - 6$ . Combining with the previous lemma,

$$(4) \quad q - 7 \leq |\Gamma_\pi| \leq q + 5,$$

for all eligible pencils except those of class (18).

3.6. The argument for class (18) is a little more intricate, but not different in substance. Calculating directly from the pencil generators, we get  $H_{2,\pi}(r, s) = h_0 r^2 + h_1 r + h_2$ , where  $h_0 = 3s^4 + 4bs^3 + 6cs^2 + 12s + 4b - c^2$ , and  $h_1, h_2$  are polynomials in  $s$  which need not be written down explicitly. Thus  $h_0$  is nonzero for all but at most 4 values of  $s$ . The  $r$ -discriminant of  $H_{2,\pi}$  is

$$\delta_\pi = h_1^2 - 4h_0h_2 = \underbrace{16(s^3 + bs^2 + cs + 1)^2}_{e_\pi(s)} \times \underbrace{[(b^2 - 3c)s^2 + (bc - 9)s + (c^2 - 3b)]}_{f_\pi(s)}.$$

In minor contrast to the earlier cases,  $e_\pi$  is the square of a cubic, and  $f_\pi$  is of degree at most 2.

We claim that the coefficients of  $s^2$  and  $s$  in  $f_\pi$  cannot vanish simultaneously. If they did, then  $b^2 = 3c, bc = 9$  and  $b, c \neq 0$  would together imply that  $b = 3\omega, c = 3\omega^{-1}$ , where  $\omega$  is a cube-root of unity. But then the polynomial  $g(T)$  from (3) has  $-\omega$  as a root, which contradicts its irreducibility. Moreover,  $f_\pi(s)$  cannot be the square of a linear form. If it were, then its  $s$ -discriminant  $-3b^2c^2 + 12b^3 + 12c^3 - 54bc + 81 = 0$ . But then  $g(T)$  cannot be irreducible, since its  $T$ -discriminant is  $b^2c^2 - 4b^3 - 4c^3 + 18bc - 27 = 0$ . (This follows from Dickson's irreducibility criterion for cubics over finite fields – see [4, Theorem 3].)

3.7. If  $f_\pi$  is a quadratic (i.e., if  $b^2 \neq 3c$ ), then apply lemma 3.3. The argument in section 3.5 also goes through, except that we may lose at most  $3 \times 3 = 9$  pairs due to singular values. If  $h_0(s) = 0$ , then we get only one  $r$ -value, hence we may lose 4 more pairs this way. Thus

$$(5) \quad q - 16 \leq |\Gamma_\pi| \leq q + 5.$$

If  $f_\pi(s)$  is linear (i.e., if  $b^2 = 3c$ ), then it is a square for  $\frac{q+1}{2}$  values of  $s$ , which gives

$$(6) \quad q - 14 \leq |\Gamma_\pi| \leq q + 1.$$

Comparing the estimates in (4), (5), (6), we deduce that

$$(7) \quad q - 16 \leq |\Gamma_\pi| \leq q + 5,$$

for *any* eligible pencil  $\pi$ .

3.8. If  $\sigma_\pi$  is the number of nonsingular members in  $\pi$ , then  $|\Psi_\pi| = \sigma_\pi(\sigma_\pi - 1)$ . According to Hirschfeld's table,  $\sigma_\pi = q - 2, q, q - 2, q + 1, q$  for classes (3), (14), (16), (18), (19) respectively. Hence  $|\Psi_\pi|$  is at least  $(q - 2)(q - 3)$ , and at most  $q(q + 1)$ . We have used the former value as the denominator of  $\mathbb{U}$ , and the latter value as the denominator of  $\mathbb{L}$ . This gives the statement of proposition 3.1. The main theorem is now completely proved.  $\square$

3.9. Assume that  $\text{char}(q) = 3$ , and reconsider the calculation in section 2.2. Since  $f(s) = (s + 1)^2$ , the quantity  $\delta$  is always a perfect square. Thus  $H_2(r, s) = 0$  has a root for any  $s$ -value, and two roots if  $s \neq -1$ . Now the same sequence of arguments shows that

$$\frac{|\Gamma|}{|\Psi|} \simeq \frac{2}{q} + O\left(\frac{1}{q^2}\right).$$

In other words, there are asymptotically twice as many (PTC)-pairs in characteristic 3.

3.10. The following proposition settles an issue raised by definition 1.1.

**Proposition 3.4.** Assume that  $q \geq 19$ . If  $(\mathcal{A}, \mathcal{B})$  is a (PTC)-pair, then there exists a nondegenerate  $\mathcal{A} \circ \mathcal{B}$  triangle.

PROOF. After applying an automorphism of  $\mathbf{P}^2$ , we can assume that  $\mathcal{B}$  is the Veronese conic

$$\{[1, t, t^2] : t \in \mathbb{F}_q\} \cup \{[0, 0, 1]\}$$

defined by the equation  $xz - y^2 = 0$ . The polar line of  $P = [\alpha, \beta, \gamma] \in \mathcal{A}$  with respect to  $\mathcal{B}$  is  $\gamma x - 2\beta y + \alpha z = 0$ . It will intersect  $\mathcal{B}$  if the polynomial  $\alpha t^2 - 2\beta t + \gamma = 0$  has a root in  $\mathbb{F}_q$ , i.e., if  $\beta^2 - \alpha\gamma$  is a square. Now choose a parametrisation  $u \rightarrow [f_0(u), f_1(u), f_2(u)]$  of  $\mathcal{A}$ , where the  $f_i(u)$  are polynomials of degree at most 2. Thus we are looking for solutions of the equation

$$(8) \quad v^2 - \underbrace{[f_1(u)^2 - f_0(u)f_2(u)]}_{g(u)} = 0,$$

where  $g(u)$  is a polynomial of degree at most 4. We need a lower estimate on the number of solutions of this equation. If  $g(u)$  is of degree 3 or 4 without repeated roots, then (8) is an affine elliptic curve, and then it has at least  $q - 1 - 2\sqrt{q}$  points by a theorem of Artin and Hasse (see [13, Ch. V]). If it has repeated roots or if  $\deg g(u) = 2$ , then it is a rational curve with at least  $q - 2$  points as long as (8) remains irreducible. If reducible, then it factors into  $v \pm \sqrt{g(u)} = 0$ , and hence must have at least  $2q$  solutions. Since a single  $u$ -value leads to at most two solutions, in any event we have at least<sup>5</sup>  $\frac{1}{2}(q - 1 - 2\sqrt{q})$  points  $P$  on  $\mathcal{A}$  from which a tangent can be drawn to  $\mathcal{B}$ .

Now a degenerate triangle involves a common tangent to  $\mathcal{A}$  and  $\mathcal{B}$ , of which there are at most 4. Hence we will have at least one nondegenerate triangle if  $\frac{1}{2}(q - 1 - 2\sqrt{q}) > 4$ , which is assured if  $q \geq 19$ .  $\square$

#### 4. A CONJECTURE

Since Poncelet's porism is true for  $n$ -gons in place of triangles, it is natural to ask whether the main theorem would generalise accordingly. Cayley's criterion for an arbitrary  $n$  involves a Hankel determinant with entries taken from the sequence  $H_2, H_3, \dots$  (see [8]). For instance, there exists a tetragon inscribed in  $C_r$  and circumscribed around  $C_s$ , if and only if

$$H_3 = s^6 - (2r + 2)s^5 + 5rs^4 - 5r^2s^2 + (2r^3 + 2r^2)s - r^3 = 0.$$

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<sup>5</sup>Although a more refined estimate is possible, the increase in complexity is not worth the effort.

The analogous conditions for pentagons and hexagons are respectively

$$\begin{vmatrix} H_2 & H_3 \\ H_3 & H_4 \end{vmatrix} = 0, \quad \text{and} \quad \begin{vmatrix} H_3 & H_4 \\ H_4 & H_5 \end{vmatrix} = 0,$$

but the corresponding polynomials are already too cumbersome to write down. I have made some computational experiments in MAPLE to count the number of root-pairs of such polynomials; they seem to support the following conjecture:

**Conjecture 4.1.** The proportion of conic pairs in  $\mathbf{P}^2(\mathbb{F}_q)$  satisfying the Poncelet  $n$ -gon condition is asymptotically equal to  $\tau_n/q$ , for some *integer* value  $\tau_n$ .

We have  $\tau_3 = 1$ , by the main theorem of this paper. Based upon experimental data, the next few values are conjectured to be:

$$\tau_4 = 3, \quad \tau_5 = 1, \quad \tau_6 = 4, \quad \tau_7 = 1, \quad \tau_8 = 6, \quad \tau_9 = 2.$$

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